Systems of Equations and Augmented Matrices

Finite Math

11 October 2018

Quiz

What possibilities are there for the number of solutions to a system of equations?

Other Types of Solutions

We now want to look at the case when the system does not have one unique solution, but is either inconsistent or is consistent but dependent.

Example

Solve the system

$$\begin{array}{rcrcr} 2x & + & 6y & = & -3 \\ x & + & 3y & = & 2 \end{array}$$

Example

Solve the system

$$\begin{array}{rcrcr} x & - & \frac{1}{2}y & = & 4 \\ -2x & + & y & = & -8 \end{array}$$

Example

Solve the system

1. $$\begin{array}{rcrcr} 5x & + & 4y & = & 4 \\ 10x & + & 8y & = & 4 \end{array}$$

2. $$\begin{array}{rcrcr} 6x & - & 5y & = & 10 \\ -12x & + & 10y & = & -20 \end{array}$$

Applications

There are a variety of applications of systems of equations. For a simple example, consider the following

Example

Dennis wants to use cottage cheese and yogurt to increase the amount of protein and calcium in his daily diet. An ounce of cottage cheese contains 3 *grams of protein and* 15 *milligrams of calcium. An ounce of yogurt contains* 1 *gram of protein and* 41 *milligrams of calcium. How many ounces of cottage cheese and yogurt should Dennis eat each day to provide exactly* 62 *grams of protein and* 760 *milligrams of calcium?*

Example

A fruit grower uses two types of fertilizer in an orange grove, brand A and brand B. Each bag of brand A contains 8 *pounds of nitrogen and* 4 *pounds of phosphoric acid. Each bag of brand B contains* 7 *pounds of nitrogen and* 6 *pounds of phosphoric acid. Tests indicate that the grove needs* 720 *pounds of nitrogen and* 500 *pounds of phosphoric acid. How many bags of each brand should be used to provide the required amounts of nitrogen and phosphoric acid?*

Solution

41 *bags of brand A and* 56 *bags of brand B.*

Matrices

Definition (Matrix)

A matrix is a rectangular array of numbers written within brackets. The entries in a matrix are called elements of the matrix.

Some examples of matrices:

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 5 & 12 & 4 \\ 0 & 1 & 8 & 3 \\ -3 & 0 & 9 & 0 \\ 7 & -9 & 22 & 10 \end{bmatrix}$$

Matrices

Definition

A matrix is called an $m \times n$ matrix if it has m rows and n columns. The expression $m \times n$ is called the size of the matrix. The numbers m and n are called the dimensions of the matrix. If $m = n$, the matrix is called a square matrix. A matrix with only 1 column is called a column matrix and a matrix with only 1 row is called a row matrix.

For example, the matrix A above is a 2×3 matrix and the matrix B is a 4×4 matrix and so B is a square matrix.

Matrices

When we write an arbitrary matrix we use the *double subscript notation*, a_{ij}, which is read as “a sub i-j”, for example, the element a_{23} is read as “a sub two-three” (not as “a sub twenty-three”); sometimes we will drop “sub” and just say “a two-three”. Here is an example arbitrary $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrices

The *principal diagonal* (or main diagonal) of a matrix is the diagonal formed by the elements $a_{11}, a_{22}, a_{33}, \ldots$. This diagonal always starts in the upper left corner, but it doesn't have to end in the bottom right.

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Augmented Matrices

In this section, we will stick with systems of 2 equations. Given a system of equations

$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & k_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & k_2 \end{array}$$

we have two matrices that we can associate to it, the *coefficient matrix*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and the *constant matrix*

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

We can also put these two matrices together and form an *augmented matrix* associated to the system

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right].$$

Augmented Matrices

Example

Find the augmented matrix associated to the system

$$\begin{array}{rcrcr} 3x & + & 4y & = & 1 \\ x & - & 2y & = & 7 \end{array}$$

Notation

We will number the rows of a matrix from top to bottom and the columns of a matrix from left to right. When referring to the i^{th} row of a matrix we write R_i (for example R_2 refers to the second row) and we use C_j to refer to the j^{th} column.

Using Augmented Matrices

Definition (Row Equivalent)

We say that two augmented matrices are row equivalent if they are augmented matrices of equivalent linear systems. We write a $\sim$ between two augmented matrices which are row equivalent.

This definition immediately leads to the following theorem

Theorem

An augmented matrix is transformed into a row-equivalent matrix by performing any of the row operations:

(a) *Two rows are interchanged* ($R_i \leftrightarrow R_j$).

(b) *A row is multiplied by a nonzero constant* ($kR_i \to R_i$).

(c) *A constant multiple of one row is added to another row* ($kR_j + R_i \to R_i$).

The arrow $\to$ is used to mean "replaces."

Solving Linear Systems Using Augmented Matrices

When solving linear systems using augmented matrices, the goal is to use row operations as needed to get a 1 for every entry on the principal diagonal and zeros everywhere else on the left side of the augmented matrix. That is, the goal is to turn it into an augmented matrix of the form

$$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$$

which corresponds to the system

$$\begin{aligned} x \qquad &= m \\ y &= n \end{aligned}$$

thus telling us that $x = m$ and $y = n$.

Example

Example

Solve the following system using an augmented matrix

$$\begin{array}{rcrcr} 3x & + & 4y & = & 1 \\ x & - & 2y & = & 7 \end{array}$$